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Methods for the Wind-Tunnel Measurement of Unsteady Nonlinear Aerodynamic Forces

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Nonlinear aerodynamic forces are a feature of many shapes and flow regimes of current interest. Well-developed methods are available for the wind-tunnel measurement of unsteady aerodynamic loads using a linearized mathematical model. Free-flight and range techniques allow a more general representation. In this paper, nonlinear wind-tunnel methods are considered involving both random and deterministic excitation. It is suggested that a pure harmonic model motion (rather than excitation) has advantages both in analysis and in experimental accuracy. This has direct application to equipment in which the model is rigidly constrained to perform a pure motion. An analogue computer validation was performed. A quasi-steady variation and the extension to multidegree of freedom are discussed.

Nomenclature

a = parameter in the analogue, Eq. (22)
A_1,A_2,A_3,A_4 = amplitudes of frequency components in $E(t)$
a_{mn},a_m,b_m etc. = coefficients used in the expansion of the nonlinear aerodynamic moment M_A
B = structural damping coefficient
C,C_1,C_2 = integration constants
$e,e_0,E(t)$ = electrical analogue signals
e_{st} = coefficients defined by Eq. (15)
$f(\theta),g(\theta,\dot\theta)$ = functions used in the description of the nonlinear moment M_A
I = model moment of inertia
$J_n(m\epsilon/\Omega)$ = Bessel function of the first kind with m and n integers
k = normalizing coefficient
K,K_1 = torsional and translational spring stiffnesses
l = characteristic model length
$M(t)$ = excitation moment
M_A = aerodynamic moment
M_b = model mass
$M_\theta,M_{\dot\theta},M_q$ etc. = aerodynamic stability derivatives
q_r = pitch rate
T = time delay
U_0 = undisturbed free stream velocity
w = translational velocity, $w = \dot z$
$W(\theta,\dot\theta)$ = joint probability density function

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z, z_0 = translational variable
α_{pq} = coefficients defined by Eq. (14)
$\delta(T)$ = delta function
ϵ = amplitude of oscillation in Eq. (17)
$\bar{\epsilon}$ = small parameter
θ, θ_0, Ω = rotational variables
θ_1 = Integration constant
μ = parameter in the analogue, Eq. (22)
ν = frequency parameter $\nu = \omega l/U_0$
τ = nondimensional time $\tau = \omega t$
Φ = phase angle
ω = frequency of oscillation

1. Introduction

A KNOWLEDGE of the steady forces and moments acting on an aircraft in flight is required for the estimation of performance in terms of range and endurance. Prediction of the unsteady forces and moments is necessary to assess the stability, control and response of a flight vehicle. In many cases, performance is paramount in dictating shape and layout since stability can be augmented artificially. It is nevertheless desirable to have as accurate a description as possible of the unsteady forces and moments to facilitate the total system design through pilot/vehicle matching or in the optimization of parameter values in automatic control loops. Design processes are well established whether the system characteristics are described on a linear or nonlinear basis. Sophisticated techniques are also available for accurately measuring aerodynamic stability derivatives, i.e., unsteady loads on a linear basis.[1,2] Free-flight and range techniques permit a more general formulation[3] since the time histories of the aerodynamic loads can be estimated from the flight path and attitudes of the projected model. At present, no similar method exists for dealing with wind-tunnel dynamic tests involving nonlinear aerodynamic characteristics. An essential preliminary to experimental testing is the development of suitable nonlinear mathematical models and little effort has been devoted to their formulation.

Nonlinear aerodynamic forces are a feature of many shapes and flow regimes currently of interest in a wide range of flight vehicles. They are generally associated with flow separation which may or may not be well behaved. For example, the leading edge separations of the slender wing planforms used on current supersonic transport designs give rise to lift increments above a constant lift-curve slope, which are vital for reasonable take-off distances. On the other hand, high-speed helicopters operating at large values of the ratio of forward speed to blade tip speed will have significant areas of the rotor disc in which the blades are stalled. Missiles, rockets, and re-entry vehicles may have partially separated flow usually at the base but sometimes at the nose if it is sufficiently blunt. Such nose separations can occur at incidences as low as 2° or 3° and in the presence of large afterbodies can produce drastic changes in both the pitch stiffness and damping derivatives.

The purpose of this paper is therefore to describe a few possible methods for wind-tunnel dynamic testing in the presence of nonlinear aerodynamic forces. The methods are described by mathematical models and are as yet untried experimentally although a small analogue study was undertaken and is described in a later section. The work also has application in the interpretation of measurements made using a linear basis even though nonlinear processes are known to be involved. It is assumed that the nonlinearities can be defined by suitable regular functions of the variables such as incidence, pitch rate, etc. The case of random loading such as might be associated with intermittant buffetting is not included. The emphasis is mainly on rigid body modes of motion in one degree of freedom although some consideration is given to multidegree-of-freedom extensions.

2. Features of Existing Dynamic Stability Measurement Techniques

The dynamic similarity parameter which must be reproduced in wind-tunnel tests is the ratio of the transport time of a fluid element to the time constant of the characteristic motion. For example, in the case of an oscillatory motion with frequency ω, the dynamic parameter is the reduced frequency or frequency parameter

$$\nu = \omega l/U_0$$

where l is a characteristic model length and U_0 is the undisturbed freestream velocity. From aircraft handling criteria requirements, full-scale short-period oscillations of about 3 rad/sec are desirable. Thus, typically, the corresponding values of ν based on tail moment arm range from about 0.05 to 0.25. In many tunnel tests U_0 is determined by the need to reproduce full-scale Mach number, and then the model frequency must increase in proportion to the inverse of the scale. It is therefore found that experimental frequencies ranging from 5 Hz to as high as 100 Hz are necessary. This relatively high forcing frequency requirement can lead to large forcing power demands and poor resolution of measured aerodynamic loads, particularly damping. Some of the special techniques used to overcome these difficulties are now briefly surveyed, for comparison with the methods later proposed for dealing with nonlinear forms.

The mathematical model used as the basis for most single-degree-of-freedom techniques is the second-order linear differential equation:

$$I\ddot{\theta} + B\dot{\theta} + K\theta - M_{\dot{\theta}}\dot{\theta} - M_\theta\theta = M(t) \tag{1}$$

where I is the moment of inertia, K is the model support stiffness, B is the structural damping coefficient and θ is taken here as model attitude relative to the tunnel axis. The form of the equation is unaltered for translational modes. $M_{\dot{\theta}}$ and M_θ are stability derivatives and $M(t)$ is the excitation.

Various forms of excitation can be used in Eq. (1). For a free oscillation [$M(t) = 0$], the equipment and instrumentation are relatively simple. Damping coefficients are measured from the decay rates of the oscillation and the aerodynamic stiffness derivative M_θ from the difference in natural frequency with and without the tunnel flow. In the majority of dynamic tests a pure harmonic excitation is used so that the motion can be described by

$$\theta = \theta_0 \sin\omega t$$

where θ_0 is the amplitude of motion. The amplitudes of $M(t)$ then become:
in phase with θ

$$(K - M_\theta - \omega^2 I)\theta_0$$

90° out of phase with θ

$$\omega(B - M_{\dot{\theta}})\theta_0$$

Without a resilient support ($K = 0$), the inertia term would predominate because of the relatively large model test frequencies and the high model density. It would therefore be difficult to obtain $M_{\dot{\theta}}$ or M_θ with any accuracy. If a spring support is used and a forcing frequency chosen so that $\omega^2 = K/I$, then both stiffness and damping derivatives can be measured readily. Alternatively, the frequency can be chosen so that the in-phase component of $M(t)$ is zero, a condition of phase resonance. The excitation may be by electromagnetic devices, in which case the amplitude of oscillation can be varied, but any nonlinearity in the system such as in the aerodynamic load will feed back to the exciter and cause distortion of the motion. If a rigid mechanical drive is used, a constant amplitude of motion is produced corresponding to the linkage eccentricity employed. The purity of the harmonic motion is maintained to a high level by the use of fly wheels on the drive motor. In this case nonlinear loads produce distortion in the transducer signal $M(t)$ rather than in the motion. This characteristic is particularly advantageous in the nonlinear analysis method described in Sec. 4.4. Surveys on the wind-tunnel techniques for the measurement of oscillatory derivatives have been made by Molyneux and Bratt and should be referred to for more detail.[1,2] Multidegree-of-freedom techniques with oscillatory motion have been used to measure both stability derivatives and aerodynamic coefficients for aeroelastic modes.[1,2,4,5]

Some advantages arise in the use of a random excitation,[6] particularly for in situ tests of components such as engine compressor blades or helicopter rotors. In Sec. 4.2, an attempt is made to extend the technique to include nonlinear effects.

3. Formulation of a Nonlinear Model

Tobak and Pearson[7] have suggested the use of an indicial function approach in order to account for the effects of the previous time histories of the variables and also of nonlinear dependence on those variables. This leads to a complicated formulation of the equations of motion both for the calculation of flight vehicle stability or response and for the analysis of wind-tunnel dynamic measurements. Some simplification is possible by assuming only a limited dependence on past history, as might be realistic for the low rates of motion (low frequency parameter) of aircraft. However, it is still necessary to specify the form of the nonlinear functional dependence before the formulation can be applied to the wind-tunnel situation.

An alternative approach is to take the expansion of the aerodynamic loads in the form of a compound series with the linear coefficients as the first terms. For example, Hopkin[8] has suggested that the aerodynamic force and torque vectors can be expanded in a series involving all the variables. The variables may be taken either in the dimensional form of translational and rotational velocities or in the nondimensional form of relative wind and attitude angles. The expansions consist of all the combinations of the variables and their time derivatives each raised to a general power, i.e. is essentially a power series. If such a formulation is used as the basis of a test technique, then the induced motion must involve all the variables. In this study the power series formulation is used for some of the methods and some reduction of complexity is sought by considering single-degree-of-freedom motions although possible extensions are discussed in Sec. 6.

4. Nonlinear Test Techniques for Single-Degree-of-Freedom Systems

4.1 Requirements

The concern is with a single degree of freedom motion which, for simplicity, is taken throughout as a rotational displacement θ such as might be used for dynamic pitch tests. The equation of motion is taken in the form

$$I\ddot{\theta} = M_A(\theta,\dot{\theta}) + M(t) \tag{2}$$

where M_A is the aerodynamic moment arising from the motion and $M(t)$ is the independent forcing term. The omission of second and higher order derivatives imposes some limitation although this is not severe in practice. In the methods described in the following sections, various types of exciting moment and nonlinearity are used. Ideally, the following characteristics are desirable.

1) The method should be as simple as possible both in practical demands and in analysis procedures. At the same time, as general a description of the nonlinearity as possible is needed to allow flexibility.

2) The method should be reducible to a linear technique with the same accuracy as existing equipment.

3) It should be possible to adapt the method for use with existing equipment.

The problem can be summarized as: a) find a suitable generalized model for $M_A(\theta,\dot{\theta})$; and b) find an excitation $M(t)$ [or alternatively, motion $\theta(t)$] which allows the systematic determination of the parameters in the model of M_A.

The methods presented in the following sections are but a few of the possible approaches. A particularly attractive alternative not considered here is the "model matching" technique used in adaptive control systems. In this method, an analogue is set up and its parameters adjusted until the response to an input signal is identical with that of the real (tunnel model/support) system subjected to the same input signal. The matching may be carried out on- or off-line.

Although simplification is achieved by considering single-degree-of-freedom tests, it is still necessary to transfer the aerodynamic moment from the tunnel axes (variable θ) to flight vehicle stability axes (variables θ, q_r, w). In fact, the expansion $M_A(\theta,\dot{\theta})$ proposed for the tunnel test arises from the corresponding stability axis expansion in such a way that

$$M_A(\theta,\dot{\theta}) = \sum_{m\,n,s,t} a_{mnst}\theta^m q_r^n w^s \dot{\omega}^t \qquad m,n,s,t \text{ integers}$$

but with

$$q_r = \dot{\theta},\ w = U_0 \sin\theta,\ \dot{\omega} = U_0\dot{\theta}\cos\theta$$

Here U_0 is the tunnel freestream velocity. This process in the linearized case gives rise to the familiar result that a fixed axis rotational test gives a measure of the combination $M_q + U_0 M_{\dot{w}}$; the term $M_{\dot{w}}$, and hence M_q, must be obtained from a separate test using the translational mode. The use of the nonlinear form means that a similar separation process is no longer possible. Rotation and translation must both be employed before complete separation is possible.

4.2 Markov Process Model

Stephens[9] has suggested the use of random excitation for nonlinear measurements. Here a different analysis is proposed although the excitation is still "white noise." The basis of this model is that the tunnel test variable $\theta(t)$ can be approximated by a Markov process obeying a fluctuation equation of the same form as Eq. (2). The method[10] is illustrated with the use of a less general form of nonlinearity than presented in Eq. (2), i.e., of the form

$$I\ddot{\theta} + B\dot{\theta} + f(\theta) = M(t) \tag{3}$$

where $B\dot{\theta} + f(\theta)$ replaces $-M_A(\theta,\dot{\theta})$ in Eq. (2). The mathematical consequence of the assumption of a Markov process is that, provided $M(t)$ is also a white noise process with zero mean, the nonstationary joint probability density $W(\theta,\dot{\theta})$ obeys the Fokker-Planck equation in the form[10]

$$\frac{\partial W}{\partial t} = \left\{\frac{1}{I} f(\theta)\frac{\partial W}{\partial \dot{\theta}} - \dot{\theta}\frac{\partial W}{\partial \theta}\right\} + \frac{B}{I}\left\{\frac{\partial}{\partial \dot{\theta}}(\dot{\theta}W) + \frac{k}{2IB}\frac{\partial^2 W}{\partial \dot{\theta}^2}\right\} \tag{4}$$

where k is the normalizing coefficient for the correlation function of $M(t)$, i.e., $\langle M(t)\rangle = 0$, $\langle M(t)\, M(t-T)\rangle = k\delta(T)$, where T is a time delay, $\delta(T)$ is a delta function and the symbol $\langle\ \rangle$ denotes a mean value. The stationary joint probability density satisfies Eq. (4) but with $\partial W/\partial t = 0$ and a solution is

$$W(\theta,\dot{\theta}) = C \exp\left[\frac{B^2}{k}\left\{-\frac{I}{B}\dot{\theta}^2 - \frac{2}{B}\int_{\theta_1}^{\theta} f(\theta)d\theta\right\}\right]$$

where C and θ_1 are integration constants. The separate probability densities for θ and $\dot{\theta}$ can be obtained by further integrations as

$$W(\theta) = C_1 \exp\left\{-\frac{2B}{k}\int_{\theta_1}^{\theta} f(\theta)d\theta\right\} \qquad (5)$$

$$W(\dot\theta) = C_2 \exp\{-IB\dot\theta^2/k\} \qquad (6)$$

In a practical application, k would be obtained from the autocorrelation function of known excitation signal $M(t)$ whereas the damping coefficient B is given by the measured distribution $W(\dot\theta)$ in Eq. (6). Equation (5) can be rearranged to give

$$f(\theta) = -(k/2B)(d/d\theta)\log_e[W(\theta)/C_1] \qquad (7)$$

so that the required nonlinear functional form $f(\theta)$ can be obtained directly from the measured probability density $W(\theta)$.

The great advantage of this method is that it is unnecessary to specify in advance the form of $f(\theta)$, i.e., it is completely general to the extent that the aerodynamic nonlinearity can be described by

$$M_A(\theta,\dot\theta) = B\dot\theta + f(\theta) \qquad (8)$$

In fact, more general forms can be used than that shown in Eq. (8). For example, solutions exist[10] to the Fokker-Planck equation which results from a Markov process fluctuation equation of the form

$$I\ddot\theta + \bar\epsilon g(\theta,\dot\theta) + f(\theta) = (\bar\epsilon)^{1/2}M(t)$$

provided the parameter $\bar\epsilon$ is small in an appropriate sense. In most practical wind-tunnel dynamic test situations the rms damping moment would be small compared with the rms value of $I\ddot\theta$ and provided a large range of θ is not required, the restrictions on $\bar\epsilon$ can be met.

The disadvantages of the method from a practical standpoint lie in the relatively large power requirements for the forcing moment and in the difficulty of providing an exciter with sufficient frequency bandwidth that the spectrum is a good approximation to that of white noise.

The major sources of error may be summarised as: 1) the real process $\theta(t)$ only approximates to a Markov process; 2) the finite sample length and analogue or digital analysis techniques will introduce errors in measured values of k and $W(\theta,\dot\theta)$; 3) the form of $f(\theta)$ in Eq. (7) involves the relatively insensitive logarithmic variation of the measured quantity. No attempt has been made to assess the magnitude of the errors.

4.3 Pure Harmonic Motion

Consideration of existing practice in Sec. 2 indicated that a resiliently mounted model enables accurate measurement of the linear damping derivatives by forcing at resonance. If the aerodynamic moment is nonlinear, the equation of motion can be represented by

$$I\ddot\theta + K\theta - M_A(\theta,\dot\theta) = M(t) \qquad (9)$$

where K is the support spring stiffness.

Probably the most practically convenient form of forcing function $M(t)$ is harmonic so that $M(t) = M_0 \sin\omega t$. Even assuming a suitable expansion for $M_A(\theta,\dot\theta)$, it is still necessary to adopt a further expansion for θ, in terms, say, of a Fourier series. The process is indeed one of seeking the forced solution to a generalized nonlinear differential equation. Such a solution has been discovered so far for only limited ranges and forms of system parameters. It is therefore more fruitful to look at Eq. (9) from the viewpoint of specifying the form of $\theta(t)$ rather than $M(t)$. If $\theta(t)$ is taken as a pure harmonic motion, $\theta(t) = \theta_0 \sin\omega t$, then the following advantages accrue.

1) With the correct choice of frequency, the spring moment exactly balances at every instant the moment equivalent to the product of the angular acceleration and the moment of inertia. The forcing moment is therefore required only to oppose the aerodynamic moment, i.e.,

$$M(t) = -M_A(\theta,\dot\theta) \qquad (10)$$

$$\theta = \theta_0 \sin\omega t \qquad (11)$$

and $\omega^2 = K/I$.

2) Using a suitable expansion for the aerodynamic moment, a method can be developed for determining systematically the expansion coefficients. To illustrate this, a power series expansion is taken in the form

$$M_A(\theta,\dot\theta) = a_{10}\theta + a_{01}\dot\theta + \sum_{m=1}\sum_{n=1} a_{mn}\theta^m\dot\theta^n \qquad (12)$$

Substitution can be made for θ from Eq. (11) and the result combined with Eq. (10) to give

$$M(t) = -a_{10}\theta_0\sin\omega t - \omega a_{01}\theta_0\cos\omega t - \sum_{m=1}\sum_{n=1} a_{mn}\omega^n\theta_0^{m+n}\sin^m\omega t\cos^n\omega t \qquad (13)$$

The forcing signal $M(t)$ can be operated on to yield a set of coefficients α_{pq} where

$$\alpha_{pq} = \frac{1}{\omega}\int_0^{2\pi} M(\tau)\sin^p\tau\cos^q\tau\, d\tau \qquad (14)$$

$\tau = \omega t$ and p,q are integers.

Substitution for $M(t)$ from Eq. (13) in Eq. (14) gives the relationships

$$\alpha_{pq} = \omega^{-(1+q)}\theta_0^{-(p+q)}\sum_s\sum_t \omega^t\theta_0^{s+t}a_{(s-p),(t-q)}\, e_{st} \qquad (15)$$

where

$$e_{st} = (2\pi/2^{s+t})s!t!/(s/2)!(s/2 + t/2)!(t/2)! = e_{ts}$$

Note that the use of the paired p,q and s,t subscripts here does not imply a tensor notation, and the summations are for even values only of $s > p$ and even values only of $t > q$.

To solve for a_{mn} it is necessary to assume finite upper limits to the summations in Eq. (15) which can then be expressed in the matrix form

$$\{\alpha_{pq}\} = [A_{pq}{}^{mn}]\{a_{mn}\}$$

where $[A_{pq}{}^{mn}]$ is the matrix of coefficients of a_{mn} in Eq. (15). The a_{mn} are then readily obtained by matrix inversion. The technique may thus be summarized: a) produce a measurable forcing moment $M(t)$ so that the model performs a pure harmonic motion at the frequency given by $\omega^2 = K/I$; b) analyze the signal $M(t)$ to produce the set of coefficients α_{pq} defined in Eq. (14); and c) with the appropriate values of ω and θ_0, set up the matrix $[A_{pq}{}^{mn}]$ from Eq. (15). Inversion gives a_{mn}.

In an assessment of the analysis error, it would be necessary to look at the conditioning of the matrix A and improve this if required. There are, in fact, four independent sets of equations for the coefficients a_{mn} corresponding to the four combinations of p,q odd and even. The complete matrix A can therefore be split into four with an accompanying decrease in the order. Thus a large number of terms can be included in the expansion of M_A before the matrix order becomes excessive. The technique lends itself to digital computation both in the matrix inversion and also in determining α_{pq}.

The implication of the excitation shaping is that the forcing mechanism must have a large band width. This could be achieved through an electromagnetic device but is more directly applicable to a rigidly constrained forcing mechanism where the harmonic motion can be maintained through the use of flywheel and a motor with adequate torque and power capability.[11] The force or moment transducer gives the required signal for analysis, provided it has sufficient bandwidth. The implementation of the method may require the use of an adaptive control process to produce the harmonic motion, in order to avoid a long tunnel running time per data point. Some limited assessment of the method described in this section was carried out in an analogue study described in Sec. 5.

4.4 Quasi-Steady Model

The range of reduced frequency or frequency parameter indicated in Sec. 2 for rigid body motions are sufficiently low that a quasi-steady approximation is reasonable for many tunnel model tests. This leads to simplification of the general process described in the preceding section and to a more straightforward reduction of data by a method closely parallel to that used in existing linear techniques.

It is now assumed that a pure harmonic motion can be produced and that the appropriate form of the equation of motion is

$$I\ddot{\theta} + K\theta + g(\theta)\dot{\theta} + f(\theta) = M(t) \quad (16)$$

Since in practice a likely motion would be one involving oscillation about a steady mean value, the form for θ is selected as

$$\theta = \theta_0 + \epsilon \sin\omega t \quad (17)$$

where $\omega^2 = K/I$.

The stages in the analysis are now as follows:

1) Assume expansions of the functions $f(\theta)$ and $g(\theta)$ to hold over a desired range Ω of the variable θ. A Fourier series expansion leads to a particularly convenient formulation

$$g(\theta) = \sum_{m=0} \left(a_m \cos\frac{m\theta}{\Omega} + b_m \sin\frac{m\theta}{\Omega} \right) \quad (18)$$

$$f(\theta) = \sum_{m=0} \left(c_m \cos\frac{m\theta}{\Omega} + d_m \sin\frac{m\theta}{\Omega} \right) \quad (19)$$

2) The amplitudes of $M(t)$ in phase and 90° out of phase with the motion can be obtained from Eqs. (16–19) as

$$\int_0^{2\pi} M(t) \sin\omega t \, d(\omega t) = -2\pi \sum_{m=0} \left(c_m \sin\frac{m\theta}{\Omega} - d_m \cos\frac{m\theta_0}{\Omega} \right) J_1\!\left(\frac{m\epsilon}{\Omega}\right) \quad (20)$$

$$\int_0^{2\pi} M(t) \cos\omega t \, d(\omega t) = \pi \sum_{m=0} \left(a_m \cos\frac{m\theta_0}{\Omega} - b_m \sin\frac{m\theta_0}{\Omega} \right) \times \left\{ J_0\left(\frac{m\epsilon}{\Omega}\right) + J_2\left(\frac{m\epsilon}{\Omega}\right) \right\} \quad (21)$$

where $J_n(m\epsilon/\Omega)$ is the Bessel function of the first kind and integral order.

3) The coefficients a_m, b_m, c_m, and d_m can now be determined from a Fourier analysis of the variation with θ_0 of the in-phase and out-of-phase components

$$\pi^2 a_m \left\{ J_0\left(\frac{m\epsilon}{\Omega}\right) + J_2\left(\frac{m\epsilon}{\Omega}\right) \right\} = \int_0^{2\pi} \left\{ \int_0^{2\pi} M(t) \cos\omega t \, d(\omega t) \right\} \cos\frac{m\theta_0}{\Omega} \, d\left(\frac{\theta_0}{\Omega}\right)$$

$$\pi^2 b_m \left\{ J_0\left(\frac{m\epsilon}{\Omega}\right) + J_2\left(\frac{m\epsilon}{\Omega}\right) \right\} = -\int_0^{2\pi} \left\{ \int_0^{2\pi} M(t) \cos\omega t \, d(\omega t) \right\} \sin\frac{m\theta_0}{\Omega} \, d\left(\frac{\theta_0}{\Omega}\right)$$

$$2\pi^2 c_m J_1\left(\frac{m\epsilon}{\Omega}\right) = -\int_0^{2\pi} \times \left\{ \int_0^{2\pi} M(t) \sin\omega t \, d(\omega t) \right\} \sin\frac{m\theta_0}{\Omega} \, d\left(\frac{\theta_0}{\Omega}\right)$$

$$2\pi^2 d_m J_1\left(\frac{m\epsilon}{\Omega}\right) = \int_0^{2\pi} \times \left\{ \int_0^{2\pi} M(t) \sin\omega t \, d(\omega t) \right\} \cos\frac{m\theta_0}{\Omega} \, d\left(\frac{\theta_0}{\Omega}\right)$$

This method is particularly appropriate to tests using rigidly constrained model motion, where the data is often presented in the form of variation of stiffness and damping derivatives with mean incidence. Provided the motion is pure and the moment (or force) signal has been filtered by a process equivalent to that on the left-hand side of Eqs. (20) and (21), then these variations can be related directly to the expansions on the right-hand side of Eqs. (20) and (21).

5. Analogue Study

In order to make some limited assessment of the practical possibility of the harmonic motion method described in Sec. 4.3, an analogue study was made. For this a second-order equation with cubic nonlinearity was simulated

$$I\ddot{\theta} + B\dot{\theta} + K\theta - M_\theta\theta - M_{\theta 3}\theta^3 - M_{\dot{\theta}}\dot{\theta} = M(t)$$

The electrical analogue equation took the form:

$$\ddot{e} + a\dot{e} + e + \mu e^3 = E(t) \quad (22)$$

The corresponding circuit was set up on a PACE TR 48 computer and the set values of a and μ compared with those obtained by the analysis method of Sec. 4.3.

The excitation signal $E(t)$ was "shaped" by a linear combination of the signals $\sin\omega t$, $\cos\omega t$, $\sin 3\omega t$ and $\cos 3\omega t$

$$E(t) = A_1 \sin\omega t + A_2 \cos\omega t + A_3 \sin 3\omega t + A_4 \cos 3\omega t \quad (23)$$

The proportion A_1, A_2, A_3, or A_4 of each was varied manually until no distortion was discernible in the Lissajou figure formed by the motion $e(t)$ and a reference signal $\sin\omega t$. The signal $E(t)$ given by Eq. (23) was used in conjunction with the method described in Sec. 4.3. A quasi-steady form was assumed and the series form of Eq. (12) was truncated to give the generalized equivalent of Eq. (10) as

$$E(t) = -a_{10}e - a_{01}\dot{e} - a_{20}e^2 - a_{30}e^3 - a_{40}e^4 \quad (24)$$

with $e = e_0 \sin t$ because the frequency was chosen with $\omega = 1$. Equations (23) and (24) are used in the same way as Eqs. (14) and (15) with successive (p,q) values of (0,0), (0,1), (1,0), (2,0), and (3,0). Five pairs of values are chosen because there are five unknown coefficients a_{01}, a_{10} to a_{40}. The selection is made so that all the coefficients are generated at least once in the simultaneous equation set given by Eq. (15). In this case, the set can be reduced to

$$4a_{20} + 3e_0^2 a_{40} = 0$$

$$e_0 a_{01} = A_2$$

$$e_0(4a_{10} + 3e_0^2 a_{30}) = 4A_1$$

$$6a_{20} + 5e_0^2 a_{40} = 0$$

$$e_0(6a_{10} + 5e_0^2 a_{30}) = 6A_1 - 2A_3$$

where e_0 is the amplitude of $e(t)$ and the frequency has been given its value of unity. Hence

$$a_{01} = A_2 e_0 = a, \quad a_{10} = (A_1 + 3A_3)/e_0$$

$$a_{20} = 0, \quad a_{30} = -4A_3/e_0^3 = \mu, \quad a_{40} = 0$$

The comparison of the set and measured values of all the coefficients a_{01},a_{10} to a_{40} agreed with their set values to within the measurement accuracy.

The analogue study described, although giving good correlation in the results, provides only a limited test of the problems associated with the practical application of the method described in Section 4.3. Since the form of the nonlinearity was known, it was possible to choose only those frequency components in the excitation strictly needed to produce the correctly shaped signal. In the general case a much larger range of harmonics would be required.

6. Extension to Multidegree-of-Freedom Testing

An indication is given in this section of a possible mathematical extension of the harmonic motion method described in Sec. 4.3 to multidegree-of-freedom modes. A two-degree-of-freedom system is used as an illustration and the method is outlined rather than described in detail.

The motions assumed are a rotation θ and a translation z of the axis of rotation. If the model center of gravity lies on the axis of rotation, the equations of motion become

$$I\ddot{\theta} + K\theta - M_A(\theta,\dot{\theta},w,\dot{w}) = M(t)$$

$$M_b\ddot{z} + K_1 z - Z_A(\theta,\dot{\theta},w,\dot{w}) = Z(t)$$

where $w = \dot{z}$, M_b is the model mass, K_1 is the structural stiffness coefficient associated with the translation. $M(t)$ and $Z(t)$ are the independent excitation moment and force respectively. To measure M_A accurately the following procedure is a logical extension.

1) Shape $M(t)$ and $Z(t)$ so that both θ and z perform harmonic motions of the form:

$$\theta = \theta_0 \sin\omega t, \quad z = z_0 \sin(\omega t + \phi), \quad \omega^2 = K/I$$

In general, the translational natural frequency will be different, $K/I \neq K_1/M_b$.

2) Change $Z(t)$ so that, although the harmonic motion is maintained at constant amplitude z_0, the phase angle ϕ can be varied over a wide range, preferably 0 to 2π.

3) Analyze $M(t)$ in a similar way to that in Sec. 4.3 to obtain the coefficients α_{pq} defined in Eq. (14). However, the α_{pq} are now dependent on ϕ. By choosing a total number of pq combinations and ϕ values appropriate to the number of coefficients chosen in the expansion of M_A, it is possible to obtain a set of simultaneous linear equations which can be solved by matrix inversion. The conditioning of the matrix depends on the values of ϕ used.

4) Z_A is analyzed by the same process as in procedures 1 to 3 but with a frequency of motion now given by $\omega^2 = K_1/M_b$ and the phase of θ measured relative to the z motion.

The method can be extended, in principle, to any number of modes because the phase angle from each additional mode can be varied to increase the number of α_{pq} measurements. However, both the complexity of the mathematical model and the practical problems of implementation involving large amplitudes of motion are severe.

7. Conclusions

Some methods have been proposed for the wind-tunnel measurement of unsteady aerodynamic loads involving a wide range of nonlinear characteristics. A limited assessment of one method (harmonic motion) was made in an analogue study. Good correlation was shown between set and measured values.

A practical feature of all the methods is the need for both the excitation mechanism and the load transducers to have large frequency band widths. The use of the technique in which the model is rigidly constrained to follow harmonic motion seems particularly advantageous in two of the proposed methods.

A single-degree-of-freedom motion has been assumed for the large part. Multidegree-of-freedom motion would be required to determine all the nonlinear cross-coupling terms and to enable the conversion of measured coefficients to stability axes. The systematic extension of the harmonic method to more than one degree of freedom is outlined.

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